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1. The log and power function review.

2. Find the three zeros of the polynomial z^3+8.

Then show $z^3+8 = (z^2-2z+4)(z+2)$.

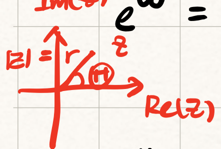
$\log z$ "inverse of e^z"

(Recall $e^{\ln x} = x \quad \forall x \in \mathbb{R}$).

$w := \log z$. and $e^w = z$.

$w = \underset{\mathrm{Re}(w)}{u} + i\underset{\mathrm{Im}(w)}{v}$ u, v are functions, valued in $\mathbb{R}$.

(If $z = x+iy$, then $u(x,y)$, $v(x,y)$.)



$e^w = e^u \cdot e^{iv} = z = re^{i\theta}$, $\underline{(-\pi < \theta \leq \pi)}$

not unique. $\theta + 2k\pi$. $k \in \mathbb{Z}$

$e^u = r$, $v = \theta + 2k\pi$, $k \in \mathbb{Z}$. $e^{i2\pi k} = 1$.

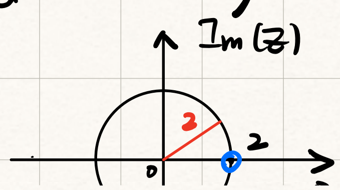
$\Rightarrow u = \ln r = \ln |z|$

$v = \arg z = \theta + 2k\pi$

$\Leftrightarrow \log z =: w = \underline{\ln |z|} + i\,\underline{\arg z}$, $z \in \mathbb{C} - \{0\}$.

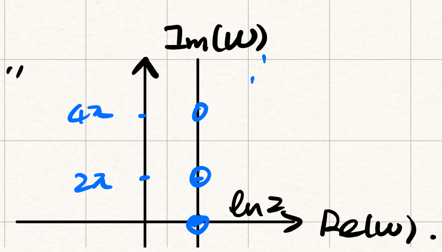
$= u + iv$.

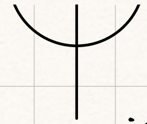
Geometrically,



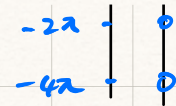
"function"

$w = \log(z)$





Re(z)



$2e^{i\theta}$, $\theta \in \mathbb{R}$.

$\log(2e^{i\theta}) = \ln 2 + i\theta$, $\theta \in \mathbb{R}$.

$z = 2e^{i\theta}$, $\theta = 2k\pi$, $k \in \mathbb{Z}$.

$\log(2e^{i\theta}) = \ln 2 + i\theta$, $\theta = 2k\pi$, $k \in \mathbb{Z}$.

One solution: Define principal value of $\log z$.

$\operatorname{Log}(z) = \ln|z| + i \operatorname{Arg} z$, $-\pi < \operatorname{Arg}(z) \le \pi$.

(single-valued function)

Recall that $\ln(xy) = \ln x + \ln y$. $\forall x, y \in \mathbb{R}$.

but it's not true for $\operatorname{Log}(z)$.

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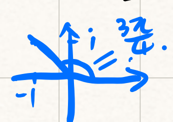
e.g. $\operatorname{Log}[(-1+i)^2] \neq 2\operatorname{Log}(-1+i)$.

$\operatorname{Log}[(-1+i)^2] = \operatorname{Log}(1-1-2i) = \ln 2 - i\frac{\pi}{2}$.

$2\operatorname{Log}(-1+i) = 2(\ln\sqrt{2} + i\frac{3\pi}{4}) = \ln 2 + i\frac{3\pi}{2}$.



Later. "branch" e.g. $\log(i^2) = 2\log(i)$

for some branch

single-valued.

Power function: z^c, $c \in \mathbb{C}$

$\underbrace{\cancel{z \cdot z \cdots z}}_{c}$ $z^c := e^{c\log(z)}$, $z \in \mathbb{C} - \{0\}$.

2. Find the three zeros of the polynomial z^3+8.

Then show $z^3+8 = (z^2-2z+4)(z+2)$.

Sol: Need z s.t. $z^3+8=0$.

$$\Rightarrow z^3 = -8, \quad n=3$$

$$-8 = 8e^{i(\pi+2k\pi)}, \quad k=0,\pm1,\cdots$$

$$z = 2\exp\left[i\left(\frac{\pi}{3}+\frac{2k\pi}{3}\right)\right], \quad k=0,1,2.$$

$$z_1 \overset{k=0}{=} 2e^{i\frac{\pi}{3}} \underset{\text{formula.}}{\overset{\text{Euler's}}{=}} 2\cos\frac{\pi}{3} + i2\sin\frac{\pi}{3}$$

$$= 1+\sqrt{3}i$$

$$z_2 \overset{k=1}{=} 2e^{i\left(\frac{\pi}{3}+\frac{2\pi}{3}\right)} = 2e^{i\pi} = -2. \qquad (e^{-i2\pi})=1$$

$$z_3 \overset{k=2}{=} 2e^{i\left(\frac{\pi}{3}+\frac{4\pi}{3}\right)} = 2e^{i\frac{5\pi}{3}} = 2e^{i\cdot\left(-\frac{\pi}{3}\right)}$$

$$\overset{\text{Euler}}{=} 2\cos\frac{\pi}{3} - i2\sin\frac{\pi}{3}.$$

$$= 1-\sqrt{3}i.$$

① Fundamental THM of Algebra.

Every nonzero single variable, degree n. polynomial with complex coefficients, counted with multiplicity has n roots.

e.g. $(x-1)^2(x-2)$

$$z^3+8 = (z-z_1)(z-z_2)(z-z_3)$$

$$= (z^2-2z+4)(z+2).$$

②. Complex conjugate root THM.

If P is a polynomial in one variable with real coefficient and $a+bi$ is a root. then so is $a-bi$.

①② $\Rightarrow$ If a real poly P has odd degree, then it has at least one real root.